Write your name and student number in the top left corner of each page

1. Determine the z-transform of the following sequences. Wherever convenient, use the properties of the z-transform to make the solution easier

   (a) $x[n] = \left(\frac{1}{2}\right)^n \mu[-n]$

   (b) $x[n] = \left(\frac{1}{3}\right)^n \mu[n] + 4^n \mu[-n-1]$

   (c) $x[n] = n\left(\frac{1}{2}\right)^n \mu[n+1]$

2. Consider the discrete-time LTI system described by the following simple difference equation:

   $y[n] = x[n] - x[n-1]$

   (a) determine the impulse respone of this system, $h[n]$, and plot $h[n]$.
   (b) determine and write a closed-form expression for the DTFT, $H(e^{i\omega})$, of $h[n]$
   (c) plot the magnitude $|H(e^{i\omega})|$ over the range $-\pi < \omega < \pi$
   (d) plot the phase of $H(e^{i\omega})$ over $-\pi < \omega < \pi$

3. Given the second order band stop filter with transfer function

   $$H_{BS}(z) = \frac{\kappa(1 - 2\beta z^{-1} + z^{-2})}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

   where α, β and γ are real constants with $|\alpha| < 1$ and $|\beta| < 1$

   (a) determine α and β so that the filter has a notch at $\omega_0 = 0.3\pi$ and a band width of 0.3π
   (b) what is the quality factor of the filter
   (c) draw the magnitude response of $H_{BS}(z)$

4. Consider the discrete-time LTI system defined by the transfer function

   $$H(z) = \frac{20 - 24z^{-1} + 20z^{-2}}{(2 - z^{-1})(2 + 2z^{-1} + z^{-2})}$$

   (a) draw the pole-zero diagram of $H(z)$
   (b) given that the impulse response $h[n]$ of this system is causal, what is the ROC
   (c) draw a block diagram which implements this transfer function in cascade form using 1st and 2nd order sections.

5. The sequence of Fibonacci number $f[n]$ is a causale sequence defined by

$$f[n] = f[n-1] + f[n-2] \ , \ n \geq 2 \ \text{ with } f[0] = 0 \text{ and } f[1] = 1$$

(a) develop an exact formula to calculate $f[n]$ directly for any n
(b) show that $f[n]$ is the impulse response of a causal LTI system described by the difference equation $y[n] = y[n-1] + y[n-2] + x[n-1]$

Solutions for the exam of 07.06.2010:

1.

a) $x[n] = \left(\frac{1}{2}\right)^n \mu[-n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \mu[-n]z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n z^{-n} = |n = -m| =$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{-m} z^m = \frac{1}{1-2z}$$

The ROC is: $|z| < \frac{1}{2}$

b)

$$x[n] = \left(\frac{1}{3}\right)^n \mu[n] + 4^n \mu[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 4^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} + \sum_{m=1}^{\infty} 4^{-m} z^m =$$

$$= \frac{1}{1-\frac{1}{3}z^{-1}} - 1 + \frac{1}{1-\frac{1}{4}z}$$

The ROC is : $\frac{1}{3} < |z| < 4$

c)

$$x[n] = n\left(\frac{1}{2}\right)^n \mu[n+1]$$

$$X(z) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n z^{-n}$$

use: $Z(ng[n]) = -z\frac{dG(z)}{dz}$ where $g[n] = \left(\frac{1}{2}\right)^n$

$$G(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z}{2z-1}$$

$$\frac{dG(z)}{dz} = \frac{2(2z-1)-4z}{(2z-1)^2}$$

$$X(z) = \frac{2z}{(2z-1)^2} = \frac{1}{2z}\frac{1}{(1-\frac{1}{2}z^{-1})^2}$$

The ROC is: $0 < |z| < \frac{1}{2}$

2.

$$y[n] = x[n] - x[n-1]$$

a)

$$h[n] = \delta[n] - \delta[n-1]$$

b)

Taking the Z transform on both sides of the equation for $y[n]$ we get:

$$Y(z) = X(z) - X(z)z^{-1}$$

then (keeping in mind that for the for determining the transfer function we need to have $x[n] = \delta[n]$), and that $Z(\delta[n]) = 1$:

$$H(z) = \frac{Y(z)}{X(z)} = 1 - \frac{1}{z}$$

To compute the DTFT, we have: $z = e^{j\omega}$, because we need to be on the unit circle:

$$H(e^{j\omega}) = 1 - \frac{1}{e^{j\omega}}$$

c)

$$|H(e^{j\omega})|^2 = \left(1 - e^{-j\omega}\right)\left(1 - e^{j\omega}\right) = 1 - e^{j\omega} - e^{-j\omega} + 1 = 2 - 2\frac{e^{j\omega} + e^{-j\omega}}{2} = 2(1 - cos\omega)$$

d)

$$\angle H(e^{j\omega}) = \arctan\left\{\frac{Im(H(e^{j\omega}))}{Re(H(e^{j\omega}))}\right\} = \arctan\left\{\frac{\sin\omega}{1-\cos\omega}\right\}$$

3.

$$H_{BS}(z) = \frac{1+\alpha}{2} \cdot \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

a)

Notch means that $|H_{BS}(e^{j\omega})| = 0$ for $\omega = 0.3\pi$, so:

$$0 = 1 - 2\beta z^{-1} + z^{-2} \Rightarrow 2\beta z^{-1} = 1 + z^{-2} \Rightarrow \beta = \frac{1+z^{-2}}{2z^{-1}} =$$

$$= \frac{z^2+1}{2z} = |z = e^{j\omega}| = \frac{e^{j2\omega}+1}{2e^{j\omega}} \left(\frac{2e^{-j\omega}}{2e^{-j\omega}}\right) = \frac{2e^{j\omega}+2e^{-j\omega}}{4} = \cos(\omega) = \cos(0.3\pi)$$

A band-stop filter has two frequencies for which $|H_{BS}(e^{j\omega})| = \frac{1}{\sqrt{2}}$. Their difference defines the stop band, and in our case it should be 0.3π radians wide. From this condition, we find that:

$$\frac{2\alpha}{(1+\alpha)^2} = \cos(0.3\pi) \Rightarrow \alpha$$

b)

The quality factor (Q - factor) is defined as: $Q = \frac{\omega_0}{\Delta\omega}$, where ω_0 is the notch frequency, and $\Delta\omega$ is the stop-band width. In our case, $\omega_0 = 0.3\pi$ and $\Delta\omega = 0.3\pi$, so we have:

$$Q = \frac{0.3\pi}{0.3\pi} = 1$$

c)

4.

$$H(z) = \frac{20 - 24z^{-1} + 20z^{-2}}{(2 - z^{-1})(2 + 2z^{-1} + z^{-2})}$$

a)

$$H(z) = \frac{20 - 24z^{-1} + 20z^{-2}}{(2 - z^{-1})(2 + 2z^{-1} + z^{-2})} = \frac{z(20z^2 - 24z + 20)}{(2z-1)(2z^2+2z+1)}$$

Solving the numerator for zeros and the denominator for the poles, we can see that we have three zeros ($z = 0, z = 0.6 \pm i0.8$) and three poles ($z = 0.5, z = -0.5 \pm i0.5$).

b)

For a causal impulse response, the ROC is all of the Z-plane outside the outermost pole circle, i.e. $|z| > \frac{1}{2}$.

c)

5.

$f[n] = f[n-1] + f[n-2], n \geq 2$ with $f[0] = 0$ and $f[1] = 1$

a)

$$f[n] = f[n-1] + f[n-2]$$

Taking the Z transform and rearanging, we get:

$$z^2 - z - 1 = 0$$

which has the solutions: $z_1 = \frac{1+\sqrt{5}}{2}$ and $z_2 = 1 - z_1 = \frac{1-\sqrt{5}}{2}$.
If we multiply the above equation by z^{n-1} we get:

$$z^{n+1} = z^n + z^{n-1}$$

Now, since z_1 and z_2 are solutions, we can express $f[n]$ as a linear combination like:

$$f[n] = az_1^n + bz_2^n$$

furthermore:

$$f[n+1] = az_1^{n+1} + bz_2^{n+1}$$

using the conditions that $f[0] = 0$ and $f[1] = 1$, we can solve the resulting system of two equations for a and b and get that $a = \frac{1}{\sqrt{5}}$ and $b = -\frac{1}{\sqrt{5}}$

So, finally we have the closed form expression:

$$f[n] = \frac{z_1^n - (1-z_1)^n}{\sqrt{5}}$$

where $z_1 = \frac{1+\sqrt{5}}{2}$

b)

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

For an impulse response, the input to the system is a delta function, i.e. in our case one sample at $n = 0$; so we can write: $x[n] = \delta[n]$. This in turn means that $\delta[n-1] = x[n-1] = 0$. Thus we have:

$$h[n] = h[n-1] + h[n-2] \equiv f[n] = f[n-1] + f[n-2]$$

Sketches accompanying the solutions:

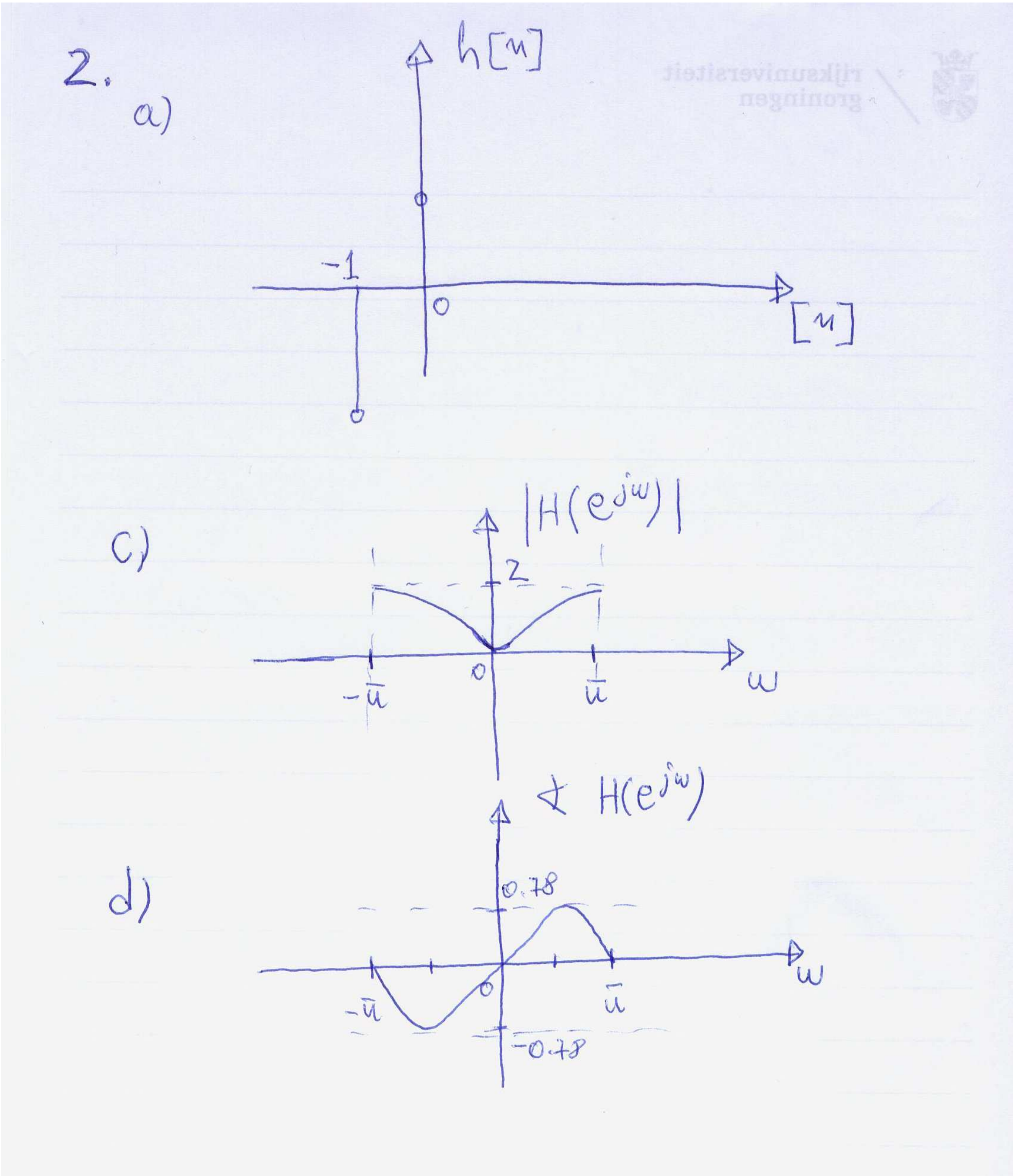

Figure 1

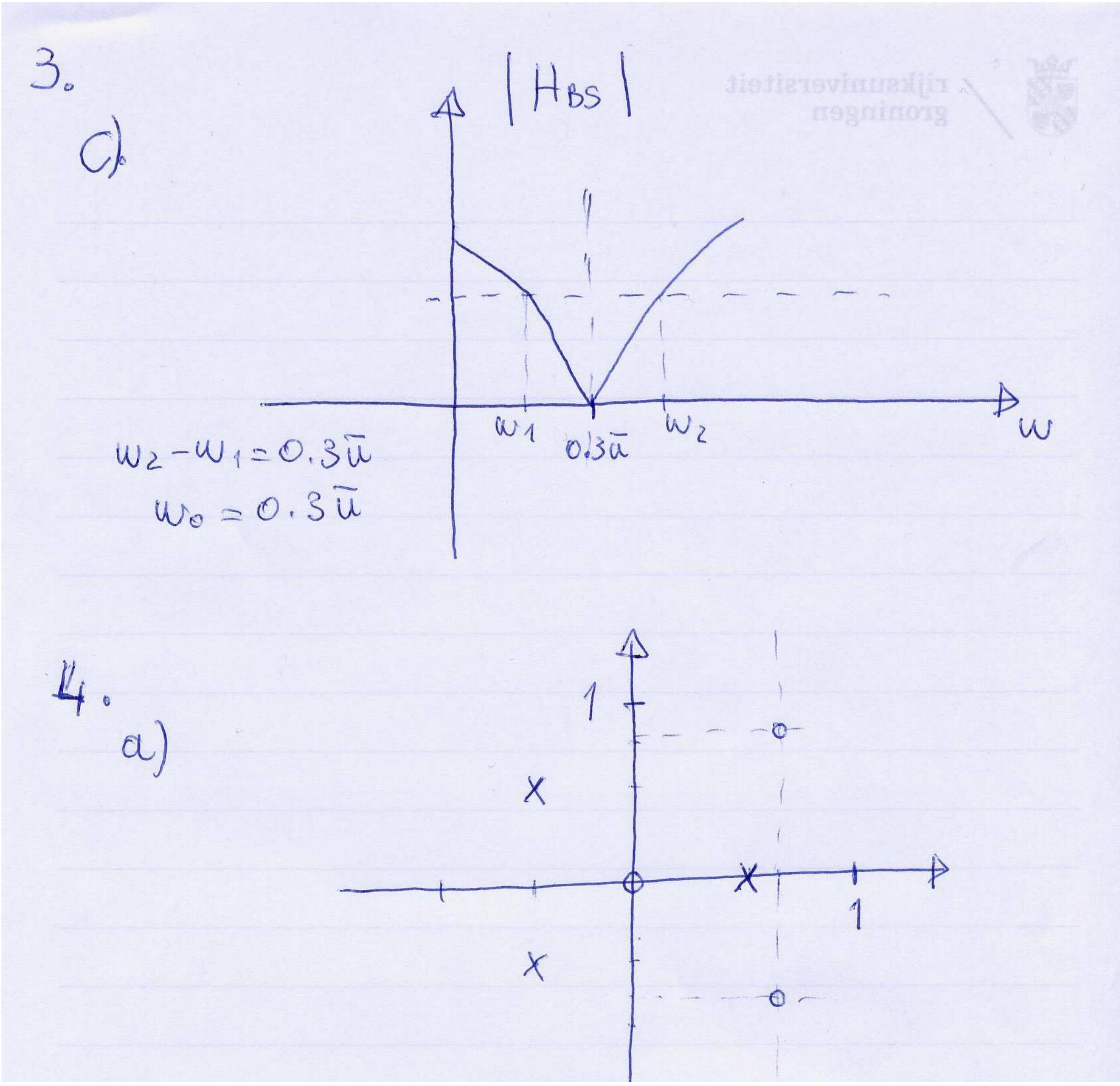

Figure 2

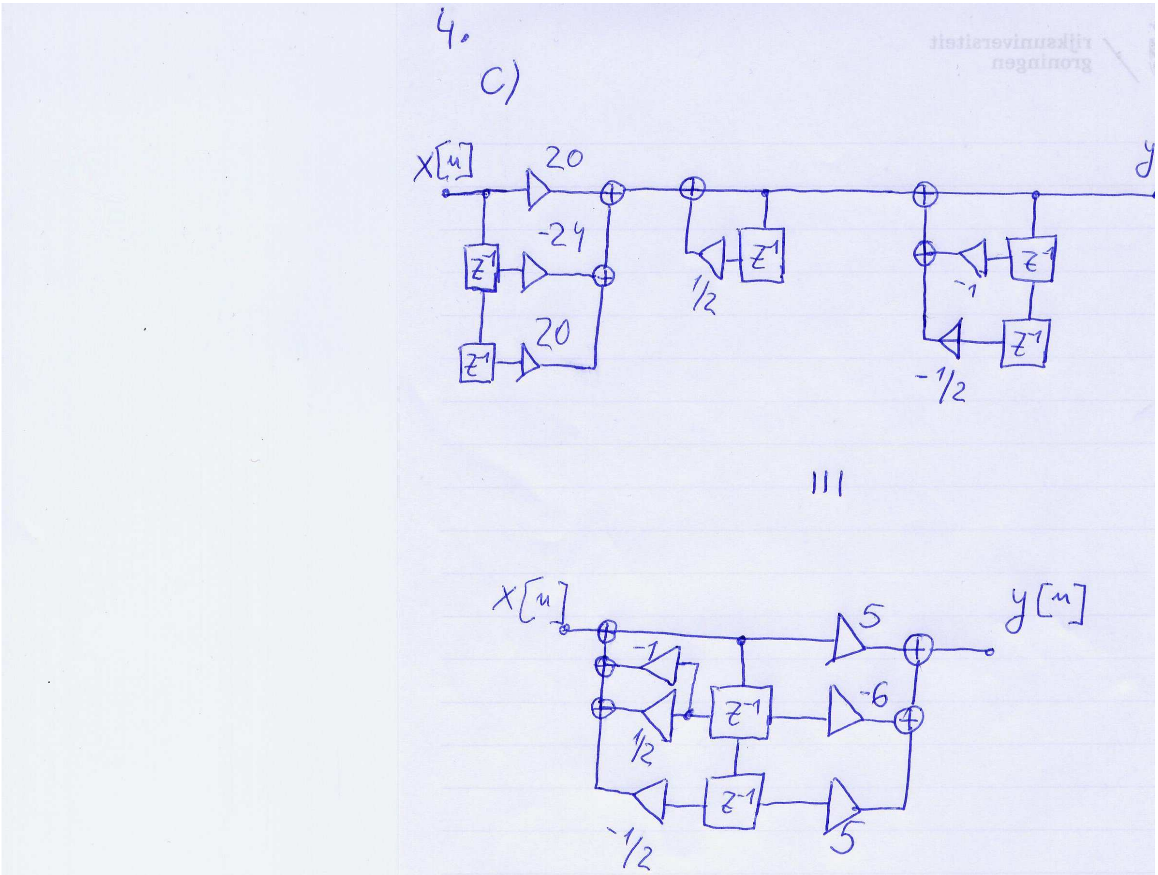

Figure 3